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Exact Resonant Frequencies for the Thickness-Twist Trapped Energy Mode in a Piezoceramic Plate

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SUMMARY
Thickness-twist modes with energy trapping in a piezoceramic plate covered by infinite strip electrodes of infinitesimal thickness are analysed. By using Fourier transforms, the linear, three-dimensional equations for a piezoceramic plate are reduced to an integral equation for the charge distribution on the electrodes. Expanding the charge density in a finite series, the lowest resonant frequency as a function of the ratio with electrodes over thickness plate is computed. The computed values are compared with the results of an approximate approach given by Holland and Eer Nisse. For small values of the mentioned ratio, considerable deviations occur.

1. Introduction

In recent years the principle of energy trapping is widely utilized in piezoelectric thickness mode resonators and filters [1]–[3]. The most simple trapped energy device is a plate of which a central region is covered by electrodes and of which the faces of the outer, surrounding region are completely free of electrodes.

The thickness-wave propagation in a plate has a fundamental cutoff frequency below which its amplitude vanishes exponentially with increasing distance to the point of excitation. The cutoff frequency of a piezoelectric plate with faces completely coated by electrodes (of finite or infinitesimal thickness) is below the one of a plate without electrodes. In addition to the possible finite thickness of the electrodes this lowering is due to the phenomenon that the behaviour of a plate with completely electroded faces differs from the behaviour of a plate without electrodes. For a piezoceramic plate with a relatively large coupling factor, the lowering due to the presence of infinitesimally thin electrodes, completely covering both faces, is considerable [1].

Hence the central part of the trapped energy plate has a cutoff frequency below the corresponding frequency in the surrounding part. Consequently energy is trapped in the central region when the plate is excited by a potential difference between the electrodes with an appropriate frequency between the cutoff frequencies of the two parts.

Energy trapping can among others be obtained in piezoceramic plates vibrating in the thickness-twist mode. This case is approximately treated by Holland and Eer Nisse [1]. An infinitely extended plate is considered with the direction of polarization in the plane of the plate. Each face is covered by an infinite strip electrode, running parallel to the polarization. Resonance spectra for some thickness-twist waves are given for a plate with an electromechanical coupling factor equal to 0.685. These spectra are obtained by determining the wave solutions in an unbounded electroded and unelectroded plate and coupling these wave solutions at the boundary of the electroded and unelectroded parts in a rough manner.

However the wave solutions are not valid near the boundary of the mentioned regions. Hence it may be expected that the spectra given in [1] are only reliable in case the disturbances due to the edge of the electrodes are relatively small, i.e. if the width of the electrodes ($2a$) is large with respect to the thickness of the plate ($2h$).

In order to obtain correct values of the resonant frequencies for all values of a/h, in this paper an exact approach of the abovementioned problem is given in case the electrodes have an infinitesimal thickness. The analysis is based on the linear, three-dimensional equations for the complete plate. Using Fourier transforms, the equations governing the plate are reduced to an integral equation for the charge distribution on the electrodes. This density is written as the sum of a term containing a squareroot singularity near the edge of the electrodes and a finite expansion in even powers. Substituting this expression into the integral equation, a system of homogeneous linear equations for the unknowns in the expansion of the charge density is obtained.

The lowest resonant frequency associated with the first thickness-twist wave is computed from this system as a function of a/h for $a/h \leqq 5$ and three values of the electromechanical coupling factor. It appears that the approximate results obtained by Holland and Eer Nisse deviate considerably from the correct values given in this paper. This deviation should be caused by the fact that the charge density has a squareroot singularity near the edge of the electrodes, which is not contained in the investigation in [1]. This singularity determines the resonant frequencies essentially for small values of the parameter a/h as follows from the results given in this paper.

2. Formulation of the Problem

We consider an infinitely extended piezoceramic plate of constant thickness $2h$ and uniformly polarized in its own plane. Figure 1 shows a cross section of a part of the plate. Cartesian coordinates (x_1, x_2, x_3) are chosen with $x_2 = \pm h$ defining the faces of the plate. The x_3-axis is in the direction of polarization. Both faces are covered by an infinitesimally thin strip electrode, occupying the regions $x_2 = \pm h$, $|x_1| \leqq a$. The faces are free of stresses; the plate is excited by a periodic potential difference between the electrodes. In the remainder the exponential time factor is omitted.

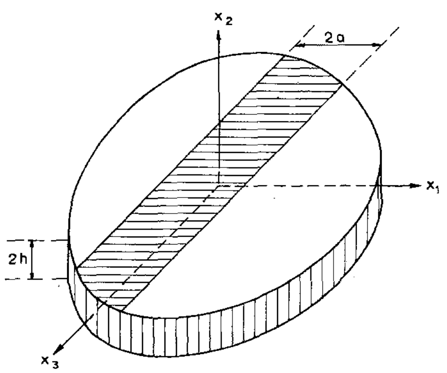

Figure 1. Cross section of a part of the plate.

A piezoceramic plate, polarized in the x_3-direction, is governed by the following constitutive relations,

$$T_{11} = {}^E c_{1111} S_{11} + {}^E c_{1122} S_{22} + {}^E c_{1133} S_{33} - e_{311} E_3 , \quad (2.1.a)$$

$$T_{22} = {}^E c_{1122} S_{11} + {}^E c_{1111} S_{22} + {}^E c_{1133} S_{33} - e_{311} E_3 , \quad (2.1.b)$$

$$T_{33} = {}^E c_{1133} (S_{11} - S_{22}) + {}^E c_{3333} S_{33} - e_{333} E_3 , \quad (2.1.c)$$

$$T_{12} = 2\,{}^E c_{1212} S_{12} , \quad (2.1.d)$$

$$T_{13} = 2\,{}^E c_{1313} S_{13} - e_{113} E_1 , \quad (2.1.e)$$

$$T_{23} = 2\,{}^E c_{1313} S_{23} - e_{113} E_2 , \quad (2.1.f)$$

$$D_1 = 2 e_{113} S_{13} + {}^S\varepsilon_{11} E_1 , \quad (2.1.g)$$

$$D_2 = 2 e_{113} S_{23} + {}^S\varepsilon_{11} E_2 , \quad (2.1.h)$$

$$D_3 = e_{311} (S_{11} + S_{22}) + e_{333} S_{33} + {}^S\varepsilon_{33} E_3 . \quad (2.1.i)$$

The stresses are denoted by the symbol T, the strains by S, the electric field by E and the electric displacement by D. ${}^{E}c_{1111} \ldots {}^{E}c_{1313}$ represent the elastic coefficients measured at constant electric field, e_{113}, e_{311} and e_{333} the piezoelectric constants and ${}^{S}\varepsilon_{11}$ and ${}^{S}\varepsilon_{33}$ the dielectric constants, measured at constant strain. The indices denote the components in the usual manner.

In addition we have the equations of motion,

$$T_{ij,i} = -\mu\omega^2 U_j , \qquad j = 1, 2, 3 , \tag{2.2}$$

the strain-displacement relations,

$$S_{ij} = \tfrac{1}{2}(U_{i,j} + U_{j,i}) , \qquad i, j = 1, 2, 3 , \tag{2.3}$$

and the Maxwell equations,

$$D_{i,i} = 0 , \tag{2.4.a}$$

$$E_i = -V_{,i} , \qquad i = 1, 2, 3 . \tag{2.4.b}$$

μ denotes the mass density of the plate, ω the circular frequency of the applied potential difference, U the geometrical displacements and V the electric potential. A comma followed by an index denotes the differentiation with respect to the coordinate indicated. In (2.2)–(2.4) the summation convention for repeated indices is employed; i and j run over 1, 2 and 3.

Since the faces are free of stresses, the mechanical boundary conditions read

$$T_{j2} = 0 , \; x_2 = \pm h , \qquad j = 1, 2, 3 . \tag{2.5}$$

Further we have

$$\begin{aligned} V &= V_0 , \quad x_2 = h , \quad |x_1| \leqq a , \\ V &= -V_0 , \quad x_2 = -h , \quad |x_1| \leqq a , \end{aligned} \tag{2.6.a}$$

$$D_2^{(1)} - D_2^{(2)} = 0 , \quad x_2 = \pm h , \quad |x_1| > a . \tag{2.6.b}$$

Here $D_2^{(1)}$ represents the electric displacement component inside the plate and $D_2^{(2)}$ this component outside the plate. We assume that outside the plate the electrostatic equations of vacuum may be applied. Hence in the regions $|x_2| > h$ the equations (2.4) are valid in addition to

$$D_i = \varepsilon_0 E_i , \qquad i = 1, 2, 3 , \tag{2.7}$$

where ε_0 is the permittivity of free space.

Now we consider thickness-twist waves propagating along x_1 and with U_1 and U_2 equal to zero. In addition we assume that the remaining displacement U_3 and the potential V are only functions of x_1 and x_2. Then the stresses T_{11}, T_{12}, T_{22}, T_{33} and the electric displacement D_3 vanish in virtue of (2.1), (2.3) and (2.4.b). The nonzero stresses and electric displacements, expressed in derivatives of U_3 and V become,

$$T_{\alpha 3} = ({}^{E}c_{1313} U_3 + e_{113} V)_{,\alpha} , \tag{2.8.a}$$

$$D_\alpha = (e_{113} U_3 - {}^{S}\varepsilon_{11} V)_{,\alpha} , \tag{2.8.b}$$

where α is 1 or 2.

For convenience we introduce a function Φ, defined by

$$\Phi = e_{113} U_3 - {}^{S}\varepsilon_{11} V . \tag{2.9}$$

Then the equations (2.8) can be written in the form

$$T_{\alpha 3} = \left({}^{D}c_{1313} U_3 - \frac{e_{113}}{{}^{S}\varepsilon_{11}} \Phi \right)_{,\alpha} , \tag{2.10.a}$$

$$D_\alpha = \Phi_{,\alpha} , \tag{2.10.b}$$

where ${}^{D}c_{1313}$ is an elastic constant measured at constant electric displacement,

$${}^{D}c_{1313} = {}^{E}c_{1313} + \frac{(e_{113})^2}{{}^{S}\varepsilon_{11}} . \tag{2.11}$$

Substituting (2.10) into (2.2) and (2.4.a), the following differential equations are obtained,

$$ {}^{D}c_{1313}\Delta U_3+\mu\omega^2 U_3 = 0\,, \tag{2.12.a}$$

$$ \Delta\Phi = 0\,. \tag{2.12.b}$$

Here Δ represents the Laplace operator with respect to x_1 and x_2.

The regions $|x_2|>h$ are governed by the equation

$$ \Delta V = 0\,. \tag{2.13}$$

The problem considered can be reduced to a problem for the quarter space $x_1 \geqq 0$, $x_2 \geqq 0$ by means of some symmetry considerations. First we observe that the plate is isotropic in the (x_1, x_2) plane. In addition the boundary conditions are symmetric with respect to the plane $x_1=0$; hence the wave solution will be symmetric in the coordinate x_1.

Let us suppose further that the solution is known, hence U_3 and V are assumed to be known functions of x_1 and x_2. With respect to the coordinates x'_1, x'_2 (fig. 2) this solution reads in virtue of the isotropy of the material in the (x_1, x_2) plane and the boundary conditions,

$$ \{-U_3,\ -V\}(x'_1, x'_2)\,. \tag{2.14}$$

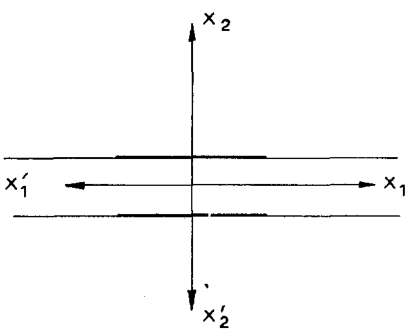

Figure 2. The coordinates x'_1 and x'_2.

The solution in the coordinates introduced above, is also obtained by applying the tensor transformation $x_1=-x'_1$, $x_2=-x'_2$ to U_3 and V, yielding

$$ \{U_3, V\}(-x'_1, -x'_2)\,. \tag{2.15}$$

Since the wave solution is even in x_1, we derive from (2.14) and (2.15),

$$ U_3 = V = 0\,, \quad x_2 = 0\,. \tag{2.16}$$

Now we can confine ourselves to the region $x_1 \geqq 0$, $x_2 \geqq 0$.

3. An Integral Expression for the Charge Distribution on the Electrodes

U_3 and V being even functions of x_1, the Fourier cosine transform may be applied. We denote the transform of a function $f(x_1, x_2)$ by $f^*(\xi, x_2)$, hence [4],

$$ f^*(\xi, x_2) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(x_1, x_2)\cos \xi x_1\, dx_1\,. \tag{3.1}$$

Multiplying both sides of (2.12) by $(2\pi)^{-\frac{1}{2}} e^{i\xi x_1}$ and integrating over the whole range of x_1, we arrive at the following ordinary differential equations,

$$ {}^{D}c_{1313}(-\xi^2 U_3^* + U^*_{3,22})+\mu\omega^2 U_3^* = 0\,, \tag{3.2.a}$$

$$ -\xi^2 \Phi^* + \Phi^*_{,22} = 0\,. \tag{3.2.b}$$

These equations enable us to assume

$$U_3^* = A_1 e^{p\xi x_2} + A_2 e^{-p\xi x_2}\,, \tag{3.3.a}$$

$$\Phi^* = e_{113}(B_1 e^{\xi x_2} + B_2 e^{-\xi x_2})\,, \tag{3.3.b}$$

where

$$p = \begin{cases} \left(1 - \dfrac{\mu\omega^2}{{}^Dc_{1313}\xi^2}\right)^{\frac{1}{2}}, & \xi \geqq \omega\left(\dfrac{\mu}{{}^Dc_{1313}}\right)^{\frac{1}{2}}, \\ i\left(\dfrac{\mu\omega^2}{{}^Dc_{1313}\xi^2} - 1\right)^{\frac{1}{2}}, & 0 < \xi < \omega\left(\dfrac{\mu}{{}^Dc_{1313}}\right)^{\frac{1}{2}}. \end{cases} \tag{3.4}$$

A_1, A_2, B_1 and B_2 are functions of ξ, determined by the boundary conditions. The factor e_{113} is introduced in order to obtain appropriate dimensions.

In virtue of the condition (2.16), $A_2 = -A_1$ and $B_2 = -B_1$, hence we can write

$$U_3^* = A \sinh p\xi x_2\,, \tag{3.5.a}$$

$$\Phi^* = e_{113} B \sinh \xi x_2\,, \tag{3.5.b}$$

where A and B are arbitrary functions of ξ for the time being. Due to (2.5) T_{23} vanishes for $x_2 = h$. Using (2.10.a), (3.1) and (3.5) this condition yields

$$Ap \cosh p\xi h - k^2 B \cosh \xi h = 0\,. \tag{3.6}$$

Here k represents an electromechanical coupling factor,

$$k = \frac{e_{113}}{({}^Dc_{1313}\,{}^S\varepsilon_{11})^{\frac{1}{2}}} \tag{3.7}$$

Applying the Fourier transform to (2.9), we obtain, using (3.5),

$$A \sinh p\xi h - B \sinh \xi h = \frac{{}^S\varepsilon_{11} V^*(\xi, h)}{e_{113}}\,. \tag{3.8}$$

Simililarly we derive from (2.10.b) and (3.5.b),

$$e_{113} B\xi \cosh \xi h = D_2^*(\xi, h)\,. \tag{3.9}$$

Combination of (3.6), (3.8) and (3.9) yields

$$D_2^*(\xi, h) = -{}^S\varepsilon_{11}\xi H(\xi) V^*(\xi, h)\,, \tag{3.10}$$

where

$$H(\xi) = \left(\frac{\sinh \xi h}{\cosh \xi h} - \frac{k^2 \sinh p\xi h}{p \cosh p\xi h}\right)^{-1}. \tag{3.11}$$

In virtue of (2.13) we have the following general solution for the region $x_1 \geqq 0$, $x_2 > h$ in terms of Fourier transforms,

$$V^* = G(\xi) e^{-|\xi| x_2}\,. \tag{3.12}$$

Hence from (2.7) and (2.4.b)

$$D_2^* = \varepsilon_0 G |\xi| e^{-|\xi| x_2} = \varepsilon_0 |\xi| V^*\,. \tag{3.13}$$

The charge distribution on the electrode $0 \leqq x_1 \leqq a$, $x_2 = h$ is denoted by $F(x_1)$ and equals the discontinuity of D_2 for $x_2 = h$. Due to (2.6.b) we have

$$F^*(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^a F(r) \cos \xi r dr\,. \tag{3.14}$$

The equations (3.10) and (3.13) lead to the expression

$$F^*(\xi) = \{\varepsilon_0 \operatorname{sign} \xi + {}^S\varepsilon_{11} H(\xi)\}\xi V^*(\xi, h)\,. \tag{3.15}$$

Applying the inverse Fourier transform, we obtain [4]

$$V(x_1, h) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty V^*(\xi, h) \cos \xi x_1 \, d\xi \,. \tag{3.16}$$

Combining (3.14), (3.15), (3.16) and the condition (2.6.a), we arrive at the following integral equation, containing the unknown $F(r)$,

$$\int_0^\infty \frac{\cos \xi x_1}{\xi\{\varepsilon_0 + {}^S\varepsilon_{11} H(\xi)\}} \int_0^a F(r) \cos \xi r \, dr \, d\xi = \frac{\pi V_0}{2}\,, \qquad 0 \leqq x_1 \leqq a \,. \tag{3.17}$$

Now we introduce the following nondimensional quantities

$$\begin{aligned} & x = \frac{x_1}{a}\,, \quad t = \frac{h}{a}\,, \quad \rho = \frac{r}{a}\,, \quad \eta = a\xi\,, \\ & G(\rho) = \frac{a}{{}^S\varepsilon_{11}} F(a\rho)\,, \quad \Omega = h\left(\frac{\mu}{{}^D c_{1313}}\right)^{\frac{1}{2}} \omega \,. \end{aligned} \tag{3.18}$$

Then (3.17) is transformed into

$$\int_0^\infty \frac{\cos \eta x}{\eta \left\{H(\eta) + \dfrac{1}{\varepsilon_r}\right\}} \int_0^1 G(\rho) \cos \eta \rho \, d\rho \, d\eta = \frac{\pi V_0}{2}\,, \qquad 0 \leqq x \leqq 1\,, \tag{3.19}$$

where

$$H(\eta) = \left(\frac{\sinh t\eta}{\cosh t\eta} - \frac{k^2 \sinh pt\eta}{p \cosh pt\eta}\right)^{-1} \tag{3.20}$$

with p defined by

$$\begin{aligned} & \left(1 - \frac{\Omega^2}{t^2 \eta^2}\right)^{\frac{1}{2}}, \qquad \eta \geqq \frac{\Omega}{t}\,, \\ & i\left(\frac{\Omega^2}{t^2 \eta^2} - 1\right)^{\frac{1}{2}}, \qquad 0 < \eta < \frac{\Omega}{t}\,. \end{aligned} \tag{3.21}$$

The quantity ε_r is a relative dielectric constant,

$$\varepsilon_r = \frac{{}^S\varepsilon_{11}}{\varepsilon_0}\,. \tag{3.22}$$

4. The Numerical Approach for Evaluating the Resonance Spectra

In order to evaluate resonant frequencies, equation (3.19) is solved with a vanishing right-hand side. Since for piezoceramic materials ε_r is a large number, the term $(\varepsilon_r)^{-1}$ occurring in the left-hand side of (3.19) is neglected. Hence we consider the integral equation

$$\int_0^\infty \frac{K(\eta) \cos \eta x}{\eta} \int_0^1 G(\rho) \cos \eta \rho \, d\rho \, d\eta = 0\,, \qquad 0 \leqq x \leqq 1 \tag{4.1}$$

where

$$K(\eta) = \{H(\eta)\}^{-1}\,. \tag{4.2}$$

It is reasonable from physical considerations that $G(\rho)$ has a squareroot singularity for $\rho = 1$. Further we know that $G(\rho)$ is an even function. Hence, in order to perform numerical calculations, $G(\rho)$ is approximated by the following series

$$a_0 \frac{2}{\pi(1-\rho^2)^{\frac{1}{2}}} + \sum_{n=1}^{N} (2n-1) a_n \rho^{2(n-1)}\,, \tag{4.3}$$

where $a_0 \ldots a_N$ are unknown constants.

The form (4.3) is substituted into (4.1) and it is required that the resulting integral equation is satisfied in $N+1$ points of the range $0 \leqq x \leqq 1$, which are denoted by x_m, $m=0 \ldots N$. Then we obtain the following system of linear, homogeneous equations for a_n,

$$\sum_{n=0}^{N} b_{mn} a_n = 0 , \qquad m = 0 \ldots N . \tag{4.4}$$

Using the well-known relation

$$\int_0^1 \frac{\cos \eta\rho}{(1-\rho^2)^{\frac{1}{2}}} d\rho = \frac{\pi}{2} J_0(\eta) , \tag{4.5}$$

and defining

$$L_n(\eta) = (2n-1) \int_0^1 \rho^{2(n-1)} \cos \eta\rho \, d\rho , \tag{4.6}$$

the coefficients b_{mn} become

$$b_{m0} = \int_0^\infty \frac{K(\eta) \cos x_m \eta}{\eta} J_0(\eta) d\eta , \tag{4.7.a}$$

$$b_{mn} = \int_0^\infty \frac{K(\eta) \cos x_m \eta}{\eta} L_n(\eta) d\eta , \qquad n = 1 \ldots N . \tag{4.7.b}$$

$J_0(\eta)$ denotes the zero-order Bessel function of the first kind. The points x_m are chosen as

$$x_m = \sin \frac{m\pi}{2N} , \qquad m = 0 \ldots N . \tag{4.8}$$

We observe that the function K depends also on Ω. Consequently the coefficients b_{mn} are functions of Ω. Resonance occur in case the determinant of the coefficients b_{mn}, denoted by $d(\Omega)$, vanishes.

In order to evaluate the lowest resonant frequency as a function of t, $d(\Omega)$ is computed for a number of values of Ω in the range

$$\Omega_e < \Omega < \Omega_u \tag{4.9}$$

and for a number of values of t. In (4.9) Ω_e denotes the first normalized cutoff frequency of the thickness-twist wave in a fully electroded, infinite plate and Ω_u this frequency in an infinite plate without electrodes. For every piezoceramic plate $\Omega_u = \frac{1}{2}\pi$ [1]; Ω_e depends on the coupling factor. By means of interpolation the zero required is obtained from the computed values of d at fixed t.

Now we discuss the computation of the coefficients b_{mn}, defined by (4.7). First we consider the quotient $K(\eta)/\eta$. From (3.20) and (3.21) we derive that the function $H(\eta)$ vanishes nowhere for $\eta \geqq \Omega/t$. $H(\eta)$ tends to a nonzero limit as $\eta \to \infty$,

$$\lim_{\eta \to \infty} H(\eta) = \frac{1}{1-k^2} . \tag{4.10}$$

Hence $K(\eta)/\eta$ is bounded in the half-infinite range $\eta \geqq \Omega/t$ and vanishes as $O(1/\eta)$ as $\eta \to \infty$.

For $0 < \eta < \Omega/t$ we derive from (3.20), (3.21) and (4.2),

$$K(\eta) = \frac{\sinh t\eta}{\cosh t\eta} - \frac{k^2 t\eta}{(\Omega^2 - t^2\eta^2)^{\frac{1}{2}}} \tan (\Omega^2 - t^2\eta^2)^{\frac{1}{2}} . \tag{4.11}$$

In virtue of the restriction (4.9) with $\Omega_u = \frac{1}{2}\pi$, we have

$$0 < (\Omega^2 - t^2\eta^2)^{\frac{1}{2}} < \tfrac{1}{2}\pi . \tag{4.12}$$

Hence $K(\eta)$ and consequently $K(\eta)/\eta$ is finite in the mentioned range. For small values of η we derive from (4.11),

$$K(\eta) = t\eta\left(1 - \frac{k^2 \tan \Omega}{\Omega}\right) + o(\eta)\,. \tag{4.13}$$

This equation yields

$$\lim_{\eta \downarrow 0} \frac{K(\eta)}{\eta} = t\left(1 - \frac{k^2 \tan \Omega}{\Omega}\right). \tag{4.14}$$

Now $K(\eta)/\eta$ is bounded for $\eta \geqq 0$.

In order to calculate the integrals in the right-hand side of (4.7), the positive η-range is divided into the subranges $0 \leqq \eta < 2$ and $\eta \geqq 2$. Since $K(\eta)/\eta$ is bounded, the integration over the first subrange can be performed easily. The functions $L_n(\eta)$ are obtained by expanding the cosine in (4.6) in a power series and integrating term by term. Then

$$L_n(\eta) = \sum_{l=0}^{\infty} \frac{(-1)^l (2n-1)\eta^{2l}}{(2l)!\,(2l+2n-1)}, \qquad n = 1 \ldots N\,. \tag{4.15}$$

By truncating the series (4.15) after a sufficient number of terms, the function $L_n(\eta)$ can be computed as accurate as desired. In virtue of the introduction of the factor $(2n-1)$ in (4.3), the functions $L_n(\eta)$ are normalized such that $L_n(0) = 1$.

When η tends to infinity, the integrands in the right-hand side of (4.7) vanish slowly and it is inefficient to perform the integrations over the second subrange directly from (4.7). Therefore some operations are applied to the mentioned integrals. In virtue of (4.2) and (4.10) we write

$$K(\eta) = 1 - k^2 + K'(\eta)\,. \tag{4.16}$$

Now $K'(\eta) \to 0$ as $\eta \to \infty$ and we have to calculate the integrals

$$\int_2^{\infty} \frac{\cos x_m \eta}{\eta} J_0(\eta)\, d\eta\,, \tag{4.17.a}$$

$$\int_2^{\infty} \frac{\cos x_m \eta}{\eta} L_n(\eta)\, d\eta\,, \tag{4.17.b}$$

$$\int_2^{\infty} \frac{K'(\eta) \cos x_m \eta}{\eta} J_0(\eta)\, d\eta\,, \tag{4.17.c}$$

$$\int_2^{\infty} \frac{K'(\eta) \cos x_m \eta}{\eta} L_n(\eta)\, d\eta\,. \tag{4.17.d}$$

In order to evaluate (4.17.a) we apply the asymptotic expansion [5],

$$J_0(\eta) \sim \left(\frac{2}{\pi\eta}\right)^{\frac{1}{2}} \left\{\cos\left(\eta - \frac{\pi}{4}\right) \cdot P(\eta) + \sin\left(\eta - \frac{\pi}{4}\right) \cdot Q(\eta)\right\} \tag{4.18}$$

where

$$P(\eta) = 1 - \frac{1^2 \cdot 3^2}{2!\,(8\eta)^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{4!\,(8\eta)^4} - \ldots \tag{4.19.a}$$

$$Q(\eta) = \frac{1^2}{1!\,8\eta} - \frac{1^2 \cdot 3^2 \cdot 5^2}{3!\,(8\eta)^3} + \ldots\,. \tag{4.19.b}$$

Using two terms of the expansion (4.19), (4.17.a) can be written as

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_2^{\infty} \frac{\cos x_m \eta}{\eta^{\frac{3}{2}}} \left\{\cos\left(\eta - \frac{\pi}{4}\right) \cdot \left(1 - \frac{9}{128\eta^2}\right) + \sin\left(\eta - \frac{\pi}{4}\right) \cdot \left(\frac{1}{8\eta} - \frac{75}{1024\eta^3}\right)\right\} d\eta$$

$$+ \int_2^{\infty} \frac{\cos x_m \eta}{\eta^{\frac{3}{2}}} R(\eta)\, d\eta\,. \tag{4.20}$$

For $R(\eta)$ we have the expression [5],

$$R(\eta) = \left(\frac{2}{\pi\eta}\right)^{\frac{1}{2}} \left\{\cos\left(\eta - \frac{\pi}{4}\right)\cdot \theta_1 \cdot \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{4!\,(8\eta)^4} + \sin\left(\eta - \frac{\pi}{4}\right)\cdot \theta_2 \cdot \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2\cdot 9^2}{5!\,(8\eta)^5}\right\}, \tag{4.21}$$

with $0 \leqq \theta_1, \theta_2 \leqq 1$.

Applying the formulae

$$\cos\alpha\,\cos\beta = \tfrac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\} \tag{4.22.a}$$

$$\cos\alpha\,\sin\beta = \tfrac{1}{2}\{\sin(\alpha+\beta) - \sin(\alpha-\beta)\} \tag{4.22.b}$$

and integrating by parts, the first integral in (4.20) can be reduced to Fresnel integrals. The latter integrals are computed by means of an Algol-procedure, available at the University of Groningen. This procedure is based on Chebyshev approximation.

In virtue of (4.21) $R(\eta)$ vanishes rapidly as $\eta \to \infty$. Hence an efficient computation of the second integral of (4.20) is possible, since we need a relative small range of integration for evaluating this integral with a certain accuracy.

From (4.6) we derive

$$L_1(\eta) = \frac{\sin\eta}{\eta}. \tag{4.23}$$

Defining

$$M_m(\rho) = \int_2^\infty \frac{\cos x_m\eta\,\sin\rho\eta}{\eta^2}\,d\eta\,, \tag{4.24}$$

the integral (4.17.b) can be denoted by $M_m(1)$ for $n=1$. Integration by parts yields for $n>1$,

$$L_n(\eta) = \frac{(2n-1)\sin\eta}{\eta} - \frac{(2n-1)(2n-2)}{\eta}\int_0^1 \rho^{2n-3}\sin\eta\rho\,d\rho\,. \tag{4.25}$$

Substituting this expression into (4.17.b) and interchanging orders of integration, we arrive at the following integral

$$(2n-1)(2n-2)\int_0^1 \rho^{2n-3}\{M_m(1) - M_m(\rho)\}\,d\rho\,. \tag{4.26}$$

The functions $M_m(\rho)$ can be expressed in the function Ci, defined by

$$Ci(u) = \int_\infty^u \frac{\cos t}{t}\,dt\,. \tag{4.27}$$

For the computation of this integral we use again an available procedure in which a Chebyshev series is applied. Evaluating $M_m(\rho)$ in this way the computation of (4.26) can be performed without any difficulty.

From (4.16) follows

$$K'(\eta) = \frac{-2e^{-2t\eta}}{1+e^{-2t\eta}} + k^2\left\{\frac{2e^{-2pt\eta}}{p(1+e^{-2pt\eta})} + 1 - \frac{1}{p}\right\} \tag{4.28}$$

for $\eta > \Omega/t$. The expression $1-1/p$ vanishes relatively slowly in comparison with the exponential functions as $\eta \to \infty$. Therefore we use the expansion

$$\frac{1}{p} \equiv \frac{1}{\left\{1 - \left(\frac{\Omega}{t\eta}\right)^2\right\}^{\frac{1}{2}}} = 1 + \frac{1}{2}\left(\frac{\Omega}{t\eta}\right)^2 + \frac{3}{8}\left(\frac{\Omega}{t\eta}\right)^4 + \frac{5}{16}\left(\frac{\Omega}{t\eta}\right)^6 + R'\,. \tag{4.29}$$

For R' we have

$$|R'| < \frac{35}{16p}\left(\frac{\Omega}{t\eta}\right)^8. \tag{4.30}$$

Now the function

$$K''(\eta) \equiv \frac{-2e^{-2t\eta}}{1+e^{-2t\eta}} + k^2 \left\{ \frac{2e^{-2pt\eta}}{p(1+e^{-2pt\eta})} - R' \right\} \tag{4.31}$$

vanishes sufficiently rapidly in order to approximate the infinite integrals

$$\int_2^\infty \frac{K''(\eta) \cos x_m \eta}{\eta} J_0(\eta) d\eta \,, \tag{4.32.a}$$

$$\int_2^\infty \frac{K''(\eta) \cos x_m \eta}{\eta} L_n(\eta) d\eta \,, \tag{4.32.b}$$

accurately by performing the integrations over a relatively small range.

To complete the calculation of (4.17.c) and (4.17.d) we have still to compute the expression

$$-k^2 \left\{ \frac{1}{2}\left(\frac{\Omega}{t}\right)^2 \int_2^\infty \frac{\cos x_m \eta}{\eta^3} f(\eta) d\eta + \frac{3}{8}\left(\frac{\Omega}{t}\right)^4 \int_2^\infty \frac{\cos x_m \eta}{\eta^5} f(\eta) d\eta + \frac{5}{16}\left(\frac{\Omega}{t}\right)^6 \int_2^\infty \frac{\cos x_m \eta}{\eta^7} f(\eta) d\eta \right\}, \tag{4.33}$$

where $f(\eta)$ equals $J_0(\eta)$ or $L_n(\eta)$, $n=1 \dots N$. The integrals in (4.33) are independent of Ω. They are computed in the same manner as (4.17.a) and (4.17.b).

The integrations discussed above are performed by means of Simpson's rule with Richardson's correction, except the computations of the Fresnel integrals and the function Ci for which an available procedure is used. If the correction term happens to come out too large, the interval is divided in two equal parts and the integration process is invoked recursively. This is done in such a way that the total amount of Richardson's corrections is slightly smaller that the prescribed, absolute tolerance. Since the integrals (4.17.a), (4.17.b) and the ones in (4.33) are independent of Ω and t, these contributions to the coefficients b_{mn} are computed first. After that the remaining integrals are calculated.

5. Numerical Results

In order to compare the approximate lowest resonant frequencies given by Holland and Eer Nisse with the corresponding frequencies obtained from the exact equations, computations are performed for $k=0.685$. The coefficients b_{mn}, defined by (4.7), are computed with an error less than 10^{-6}. For N we take the value 7. It appears that we are now ensured of 3 significant digits in the ultimate results. The computed resonant frequencies are represented by the second column of the following table. Here Ω_r denotes the first, positive zero of $d(\Omega)$, $\Omega_e=0.3811\,\pi$ and $\Omega_u=0.5\,\pi$.

To show the influence of the square-root singularity of the charge density on the resonant frequencies for large values of t, also computations are performed in case $G(\rho)$ is only ap-

TABLE 1

t	$\frac{\Omega_r-\Omega_e}{\Omega_u-\Omega_e}$	$\frac{\Omega_r-\Omega_e}{\Omega_u-\Omega_e}$
0.2	0.038	
0.5	0.144	
1	0.286	
2	0.451	0.424
5	0.633	0.629
10	0.726	0.726
20	0.788	0.788
100	0.873	0.873

proximated by the first term of the series (4.3) and the integral equation is satisfied at $x=0$. The resonant frequencies obtained in this way are given in the third column for $t \geqq 2$. For $t \geqq 10$ the results in the second and third column agree in three digits.

For large values of t the resonant frequencies can be represented by

$$\frac{-0.686}{t^{\frac{1}{2}}} + 0.942\,. \tag{5.1}$$

Hence the quotient $(\Omega_r - \Omega_e)/(\Omega_u - \Omega_e)$ tends to 0.942 as $t \to \infty$ and not to 1 as suggested in [1].

In figure 3 the computed resonant frequencies and the approximate ones, given in [1], are plotted as a function of $t^{-1} = a/h$. We observe that in addition to the mentioned deviation for $t^{-1} \to 0$ also relatively large deviations occur for values of t of $O(1)$.

In order to investigate the influence of the coupling factor also calculations are performed for some other, arbitrarily chosen values of k ($k=0.548$ and $k=0.800$). In figure 4 a plot of the

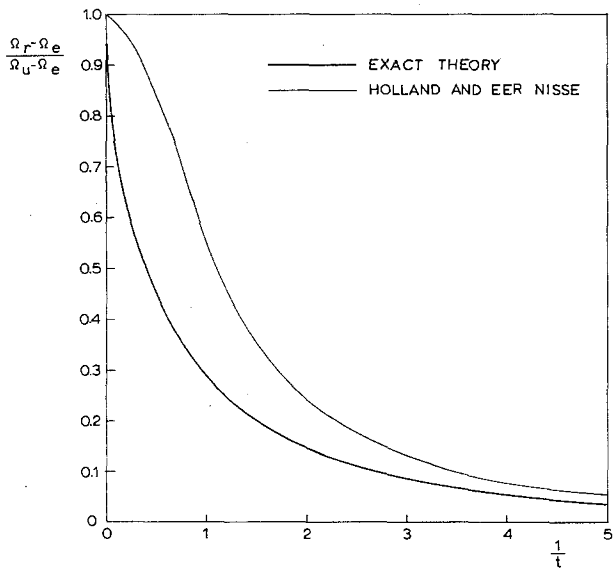

Figure 3. Correct and approximate resonant frequencies for $k=0.685$.

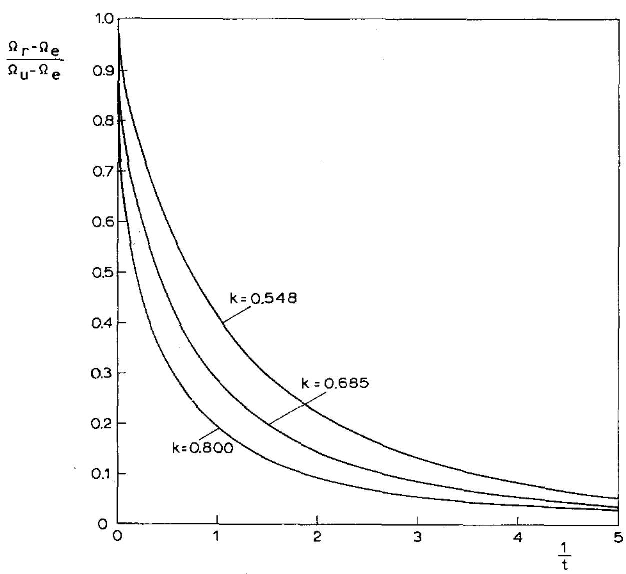

Figure 4. Resonant frequencies for three values of k.

lowest resonant frequencies for three values of k are given. For $k=0.548$ they can be represented by

$$\frac{-0.439}{t^{\frac{1}{2}}} + 0.977 \tag{5.2}$$

as $t \to \infty$ and for $k=0.800$ by

$$\frac{-0.919}{t^{\frac{1}{2}}} + 0.869\,. \tag{5.3}$$

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